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Boson stars with large self-interaction in (2+1) dimensions: an exact solution

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Abstract: An exact solution for a nonrotating boson star in $(2+1)$ dimensional gravity with a negative cosmological constant is found. The relations among mass, particle number, and radius of the $(2+1)$ dimensional boson star are shown.

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Self-gravitating systems have been investigated in various situations. Boson stars ([1, 2] for reviews) have a very simple constituent, a complex scalar field which is bound by gravitational attraction. Thus the boson star provides us with the simplest model of relativistic stars.

The solutions for relativistic boson stars are only numerically obtained in four dimensions. In $(2+1)$ dimensions, static equilibrium configurations have been argued [3] in Einstein gravity with a negative cosmological constant.

In this paper, we obtain an exact solution for a nonrotating boson star in $(2+1)$ dimensional gravity with a negative cosmological constant. We consider that the scalar field has a strong self-interaction. An infinitely large self-interaction term in the model leads to much simplications as in the $(3+1)$ dimensional case [4]. The study of exact solutions will lead to a new aspect of gravitating systems and clarify the similarity and/or the difference among the other dimensional cases.

We consider a complex scalar field with mass m and a quartic self-coupling constant λ. The action for the scalar field coupled to gravity can be written down as

$$S = \int d^3x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R+2C\right) - \left|\nabla_\mu\varphi\right|^2 - m^2\left|\varphi\right|^2 - \frac{\lambda}{2}\left|\varphi\right|^4\right], \tag{1}$$

where R is the scalar curvature and the positive constant C stands for the (negative) cosmological constant. G is the Newton constant.

Varying the action (1) with respect to the scalar field and the metric yields equations of motion. The equation of motion for the scalar field is

$$\nabla^2\varphi - m^2\varphi - \lambda\left|\varphi\right|^2\varphi = 0\,, \tag{2}$$

while the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left[2\mathrm{Re}\left(\nabla_\mu\varphi^*\nabla_\nu\varphi - \frac{1}{2}\left|\nabla\varphi\right|^2 g_{\mu\nu}\right) - m^2\left|\varphi\right|^2 g_{\mu\nu} - \frac{\lambda}{2}\left|\varphi\right|^4 g_{\mu\nu}\right] + Cg_{\mu\nu}\,. \tag{3}$$

We assume the 3D metric for a static circularly symmetric spacetime as

$$ds^2 = -\alpha^2(r)dt^2 + \beta^2(r)dr^2 + \frac{\gamma^2(r)}{\gamma_0^2}d\theta^2\,, \tag{4}$$

where α, β, and γ are functions of the radial coordinate r only, while γ_0 is a constant. Because of the coordinate invariance with respect to r, we have a freedom in choosing the function $\beta(r)$. We will use this residual "gauge choice" later.

We also assume that the complex scalar has a phase which is linear in the temporal coordinate:

$$\varphi = e^{-i\omega t}\varphi(r)\,, \tag{5}$$

where $\varphi(r)$ is a function of the radial coordinate r only and ω is a constant.

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Then the equations (2) and (3) are written as

$$\frac{\omega^2}{\alpha^2}\varphi + \frac{1}{\alpha\beta\gamma}\frac{d}{dr}\left(\frac{\alpha\gamma}{\beta}\frac{d\varphi}{dr}\right) - m^2\varphi - \lambda\varphi^3 = 0, \tag{6}$$

$$\frac{1}{\beta\gamma}\frac{d}{dr}\left(\frac{1}{\beta}\frac{d\gamma}{dr}\right) = 8\pi G\left[-\frac{1}{\beta^2}\left(\frac{d\varphi}{dr}\right)^2 - \frac{\omega^2}{\alpha^2}\varphi^2 - m^2\varphi^2 - \frac{\lambda}{2}\varphi^4\right] + C\,, \tag{7}$$

$$\frac{1}{\alpha\beta^2\gamma}\frac{d\alpha}{dr}\frac{d\gamma}{dr} = 8\pi G\left[+\frac{1}{\beta^2}\left(\frac{d\varphi}{dr}\right)^2 + \frac{\omega^2}{\alpha^2}\varphi^2 - m^2\varphi^2 - \frac{\lambda}{2}\varphi^4\right] + C\,, \tag{8}$$

$$\frac{1}{\alpha\beta}\frac{d}{dr}\left(\frac{1}{\beta}\frac{d\alpha}{dr}\right) = 8\pi G\left[-\frac{1}{\beta^2}\left(\frac{d\varphi}{dr}\right)^2 + \frac{\omega^2}{\alpha^2}\varphi^2 - m^2\varphi^2 - \frac{\lambda}{2}\varphi^4\right] + C\,. \tag{9}$$

For convenience, we rescale the variables as

$$\tilde{r} = mr, \quad \tilde{\varphi} = \sqrt{8\pi G}\varphi, \quad \Lambda = \frac{\lambda}{8\pi G m^2}\,, \quad \tilde{\omega} = \omega/m\,, \quad \textit{and} \quad \tilde{C} = C/m^2\,. \tag{10}$$

Using these variables, we can rewrite the equations as

$$\frac{\tilde{\omega}^2}{\alpha^2}\tilde{\varphi} + \frac{1}{\alpha\beta\gamma}\frac{d}{d\tilde{r}}\left(\frac{\alpha\gamma}{\beta}\frac{d\tilde{\varphi}}{d\tilde{r}}\right) - \tilde{\varphi} - \Lambda\tilde{\varphi}^3 = 0\,, \tag{11}$$

$$\frac{1}{\beta\gamma}\frac{d}{d\tilde{r}}\left(\frac{1}{\beta}\frac{d\gamma}{d\tilde{r}}\right) = -\frac{1}{\beta^2}\left(\frac{d\tilde{\varphi}}{d\tilde{r}}\right)^2 - \frac{\tilde{\omega}^2}{\alpha^2}\tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2}\tilde{\varphi}^4 + \tilde{C}, \tag{12}$$

$$\frac{1}{\alpha\beta^2\gamma}\frac{d\alpha}{d\tilde{r}}\frac{d\gamma}{d\tilde{r}} = +\frac{1}{\beta^2}\left(\frac{d\tilde{\varphi}}{d\tilde{r}}\right)^2 + \frac{\tilde{\omega}^2}{\alpha^2}\tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2}\tilde{\varphi}^4 + \tilde{C}\,, \tag{13}$$

$$\frac{1}{\alpha\beta}\frac{d}{d\tilde{r}}\left(\frac{1}{\beta}\frac{d\alpha}{d\tilde{r}}\right) = -\frac{1}{\beta^2}\left(\frac{d\tilde{\varphi}}{d\tilde{r}}\right)^2 + \frac{\tilde{\omega}^2}{\alpha^2}\tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2}\tilde{\varphi}^4 + \tilde{C}\,. \tag{14}$$

Next, we rescale the variables again, to study the limit of large self-inteaction. New variables are:

$$r_* = \tilde{r}/\sqrt{\Lambda}\,, \quad \varphi_* = \sqrt{\Lambda}\tilde{\varphi}\,, \quad \textit{and} \quad C_* = \Lambda\tilde{C}\,. \tag{15}$$

Then the equations can be written as

$$\frac{\tilde{\omega}^2}{\alpha^2}\varphi_* + \frac{1}{\Lambda}\frac{1}{\alpha\beta\gamma}\left(\frac{\alpha\gamma}{\beta}\varphi_*'\right)' - \varphi_* - \varphi_*^3 = 0\,, \tag{16}$$

$$\frac{1}{\beta\gamma}\left(\frac{\gamma'}{\beta}\right)' = -\frac{1}{\Lambda}\frac{1}{\beta^2}\left(\varphi_*'\right)^2 - \frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,, \tag{17}$$

$$\frac{1}{\beta^2}\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} = +\frac{1}{\Lambda}\frac{1}{\beta^2}\left(\varphi_*'\right)^2 + \frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,, \tag{18}$$

$$\frac{1}{\alpha\beta}\left(\frac{\alpha'}{\beta}\right)' = -\frac{1}{\Lambda}\frac{1}{\beta^2}\left(\varphi_*'\right)^2 + \frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,, \tag{19}$$

where $'$ denotes the derivative with respect to r_*.

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For the limit of large self-coupling, $\Lambda \to \infty$, these equations will be reduced to[1]

$$\frac{\tilde{\omega}^2}{\alpha^2}\varphi_* - \varphi_* - \varphi_*^3 = 0\,, \tag{20}$$

$$\frac{1}{\beta\gamma}\left(\frac{\gamma'}{\beta}\right)' = -\frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,, \tag{21}$$

$$\frac{1}{\beta^2}\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} = +\frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,, \tag{22}$$

$$\frac{1}{\alpha\beta}\left(\frac{\alpha'}{\beta}\right)' = +\frac{\tilde{\omega}^2}{\alpha^2}\varphi_*^2 - \varphi_*^2 - \frac{1}{2}\varphi_*^4 + C_*\,. \tag{23}$$

Now, let us solve the set of an algebraic equation and differential equations (20-23). From Eqs. (22) and (23), we find

$$\left(\frac{\alpha'}{\beta\gamma}\right)' = 0\,. \tag{24}$$

To solve this, we use the residual gauge choice on β. For simplicity, we take

$$\alpha\beta\gamma = r_*\,. \tag{25}$$

Then Eq. (24) is solved as

$$\alpha^2(r) = A\left(r_*^2 + r_0^2\right), \tag{26}$$

where A and r_0 are integration constants. Since the constant scale factor of α can be absorbed in the rescaling of t, we can fix the value $A = 1$. Then, for the later use, we notice that

$$\frac{d\alpha^2}{dr_*^2} = 1\,. \tag{27}$$

From Eqs. (20), (22) and (26), we find[2]

$$\frac{\gamma\gamma'}{r_*} = \frac{1}{2}\varphi_*^4 + C_*\,. \tag{28}$$

Here using Eq. (27), we obtain

$$\frac{d\gamma^2}{d\alpha^2} = \frac{1}{2}\varphi_*^4 + C_*\,. \tag{29}$$

We have to solve Eq. (29) by using the solution for φ_*. Eq. (20) can be solved easily. The interior solution for φ_* is

$$\varphi_*^2 = \frac{\tilde{\omega}^2}{\alpha^2} - 1\,. \tag{30}$$

[1]For a finite C_*, the actual value of $C = m^2 C_*/\Lambda$ becomes infinitely small if the limit $\Lambda = \infty$ is taken literally. We can however interpret the limit as an approximation of a *large* self-coupling, and Λ is not simply taken as a mathematical infinity.

[2]This equation (28) is consistent with the other equations (20) and (21).

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This describes the configuration of the boson field inside a boson star. On the other hand, outside the boson star,

$$\varphi_*^2 = 0\,. \tag{31}$$

Before solving the interior and exterior solution for the metric, we rewrite the radial line element as

$$\beta dr = \sqrt{\frac{\Lambda}{m^2}}\beta dr_* = \sqrt{\frac{\Lambda}{m^2}}\frac{r_*}{\alpha\gamma}\frac{dr_*}{d\gamma}d\gamma = \sqrt{\frac{\Lambda}{m^2}}\frac{1}{\alpha}\frac{1}{\frac{1}{2}\varphi_*^4 + C_*}d\gamma\,. \tag{32}$$

Consequently, we get

$$\beta^2 dr^2 = \frac{\Lambda}{m^2}\frac{1}{\alpha^2}\frac{1}{\left(\frac{1}{2}\varphi_*^4 + C_*\right)^2}d\gamma^2\,. \tag{33}$$

First, we solve the exterior solution. Since $\varphi_* = 0$ outside the boson star, the solution of Eq. (29) is found to be

$$\alpha^2 = \frac{1}{C_*}\gamma^2 - B\,, \tag{34}$$

where B is an integration constant.

The full line element is obtained, by substituting Eqs. (25), (33) and (34) into (4), as

$$ds^2 = -\frac{1}{C_*^2}\left(C_*\gamma^2 - C_*^2 B\right)dt^2 + \frac{\Lambda}{m^2}\frac{1}{C_*\gamma^2 - C_*^2 B}d\gamma^2 + \frac{\gamma^2}{\gamma_0^2}d\theta^2\,. \tag{35}$$

Here we must remember that $C_* = C\Lambda/m^2$. We further introduce a new radial coordinate R defined as

$$R = \sqrt{\frac{\Lambda}{m^2}}\gamma\,, \tag{36}$$

and set $\gamma_0 = \sqrt{m^2/\Lambda}$. Then we find

$$ds^2 = -\frac{1}{C_*^2}\left(CR^2 - C_*^2 B\right)dt^2 + \frac{1}{CR^2 - C_*^2 B}dR^2 + R^2 d\theta^2\,. \tag{37}$$

After rescaling $\frac{1}{C_*}dt \to dt$, we find the metric is precisely the same as the well-known BTZ vacuum solution [5]. Thus we identify the BTZ mass M_o *of the boson star* with

$$8GM_o = C_*^2 B\,. \tag{38}$$

Here we turn to solving the interior solution. From Eqs. (29) and (30), we obtain a differential equation:

$$\frac{d\gamma^2}{d\alpha^2} = \frac{1}{2}\left(\frac{\tilde{\omega}^2}{\alpha^2} - 1\right)^2 + C_*\,. \tag{39}$$

The solution of this equation is given by

$$\gamma^2 = -\frac{1}{2}\frac{\tilde{\omega}^4}{\alpha^2} - \tilde{\omega}^2 \ln\frac{\alpha^2}{\tilde{\omega}^2} + \frac{1}{2}\alpha^2 + C_*\left(\alpha^2 + D\right), \tag{40}$$

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where D is a constant. The value of D is determined from the condition at the boundary of the boson star, where $\varphi_* = 0$. One can find the condition for a smooth connection of the exterior and interior solutions (from Eqs. (33), (34) and (38)):[3]

$$C_*\gamma^2 - 8GM_o\Big|_{\alpha^2=\tilde{\omega}^2} = \alpha^2\left(\frac{1}{2}\varphi_*^4 + C_*\right)^2\Bigg|_{\alpha^2=\tilde{\omega}^2} . \tag{41}$$

From Eqs. (41) and (40), we get

$$C_*^2 D = 8GM_o\,. \tag{42}$$

What does the metric at the center of the boson star look like? As seen in Eq. (37), the origin is located at $R = 0$, or $\gamma = 0$. Therefore $\alpha_0 = \alpha(\gamma = 0)$ satisfies the following equation:

$$-\frac{1}{2}\frac{\tilde{\omega}^4}{\alpha_0^2} - \tilde{\omega}^2 \ln\frac{\alpha_0^2}{\tilde{\omega}^2} + \frac{1}{2}\alpha_0^2 + C_*\alpha_0^2 + \frac{1}{C_*}8GM_o = 0\,. \tag{43}$$

Thus the metric at the center of the boson star is written by using α_0 as

$$ds^2(R \approx 0) \approx -\alpha_0^2 dt^2 + \frac{1}{\alpha_0^2}\frac{1}{\left(\frac{1}{2}\varphi_{*0}^4 + C_*\right)^2}dR^2 + R^2 d\theta^2\,, \tag{44}$$

where $\varphi_{*0}^2 = \frac{\tilde{\omega}^2}{\alpha_0^2} - 1$ is the value of φ_*^2 at the center of the star. We can give an arbitrary value for φ_{*0}^2 by choosing the value of $\tilde{\omega}$. Thus we can take φ_{*0}^2 as a free parameter which characterizes a solution for a boson star, instead of $\tilde{\omega}$.

In higher dimension more than three, we can fix the natural value of the metric component at the center. In three dimensions, we cannot take a definite choice, because there is no *local* definition of mass; an additional constant is allowed because it does not lead to a curvature singularity directly, unlike the higher-dimensional cases. This is also the reason why an arbitrary constant remains in the study of Cruz and Zanelli [3] on relativistic stars in $(2+1)$ dimensions.

Nevertheless, we can choose a plausible choice of the metric at the center,[4] in order to give the definite properties of the boson star; if one wishes to know the general case, one need only loose the constraint in the following analysis.

We choose the following condition:

$$\alpha_0^2\left(\frac{1}{2}\varphi_{*0}^4 + C_*\right)^2 = 1\,. \tag{45}$$

According to this choice, the spatial metric at the origin looks like the one with no conical singularity. The condition (45) determines the value of α_0, as a function of φ_{*0}.

This choice for the boundary condition makes the mass of the scalar boson definite. In other words, if we take other values, the parameter m does not represent the mass

[3]The first derivative is automatically connected when this condition (41) is satisfied.

[4]The conical sigularity at the origin is not *a priori* governed by the *differential* equations because only the boundary condition is concerned. This is even true for a vacuum case.

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of the excitation of the boson field φ near the origin; the value must be rescaled. In the present case, since there is no asymptotically flat spacetime region, the identification of mass parameter in the most interior region is reasonable.

We obtain the relation between the BTZ mass of the boson star and φ_{*0} from Eqs. (43) and (45):

$$\frac{8GM_o}{C_*} = \frac{1}{\left(\frac{1}{2}\varphi_{*0}^4 + C_*\right)^2}\left[\frac{1}{2}\varphi_{*0}^4 + \varphi_{*0}^2 - \left(\varphi_{*0}^2+1\right)\ln\left(\varphi_{*0}^2+1\right) - C_*\right]. \tag{46}$$

The value of BTZ mass M_o is negative for small φ_{*0}. In particular, as φ_{*0} goes to zero, M_o approaches $-1/(8G)$. One can take

$$M_i = M_o + \frac{1}{8G}\,, \tag{47}$$

as a definition of the intrinsic mass M_i, we call here. When φ_{*0} vanishes, M_i becomes zero, and spacetime becomes a usual anti-de Sitter spacetime everywhere, i.e.,

$$ds^2 = -\left(1+CR^2\right)dt^2 + \frac{dR^2}{1+CR^2} + R^2 d\theta^2\,, \tag{48}$$

where the temporal coordinate has been appropriately normalized. Then we have so-called AdS vacuum.[5]

The total particle number N is given by

$$N = \int d^2x\sqrt{-g}\left|g^{tt}\right| i\left(\varphi\partial_t\varphi^* - \varphi^*\partial_t\varphi\right). \tag{49}$$

Therefore we can calculate the particle number of the boson star by making use of the above solution. It is given by

$$\begin{aligned} N &= 2\pi\int\frac{2m\tilde{\omega}}{8\pi G\Lambda}\varphi_*^2\frac{1}{\alpha^2}\frac{\alpha\beta\gamma}{\gamma_0}dr \\ &= \frac{m\tilde{\omega}}{2G\Lambda\gamma_0}\int\left(\frac{\tilde{\omega}^2}{\alpha^2}-1\right)\frac{1}{\alpha^2}r_*\sqrt{\frac{\Lambda}{m^2}}dr_* \\ &= \frac{m\tilde{\omega}}{4G\Lambda\gamma_0}\sqrt{\frac{\Lambda}{m^2}}\int_{\alpha_0^2}^{\tilde{\omega}^2}\left(\frac{\tilde{\omega}^2}{\alpha^2}-1\right)\frac{1}{\alpha^2}d\alpha^2 \\ &= \frac{m}{4G\Lambda\gamma_0}\sqrt{\frac{\Lambda}{m^2}}\alpha_0\frac{\tilde{\omega}}{\alpha_0}\left(\frac{\tilde{\omega}^2}{\alpha_0^2}-1-\ln\frac{\tilde{\omega}^2}{\alpha_0^2}\right) \\ &= \frac{1}{4Gm}\frac{\sqrt{\varphi_{*0}^2+1}\left[\varphi_{*0}^2-\ln\left(\varphi_{*0}^2+1\right)\right]}{\frac{1}{2}\varphi_{*0}^4+C_*}\,. \end{aligned} \tag{50}$$

The intrinsic mass and particle number of the boson star as functions of φ_{*0} are plotted in Fig. 1.

[5]Note that, in the presence of matter, the limit of "no matter" cannot yields so-called black hole vacuum, which is given by BTZ geometry in the limit $M_o = 0$ [3].

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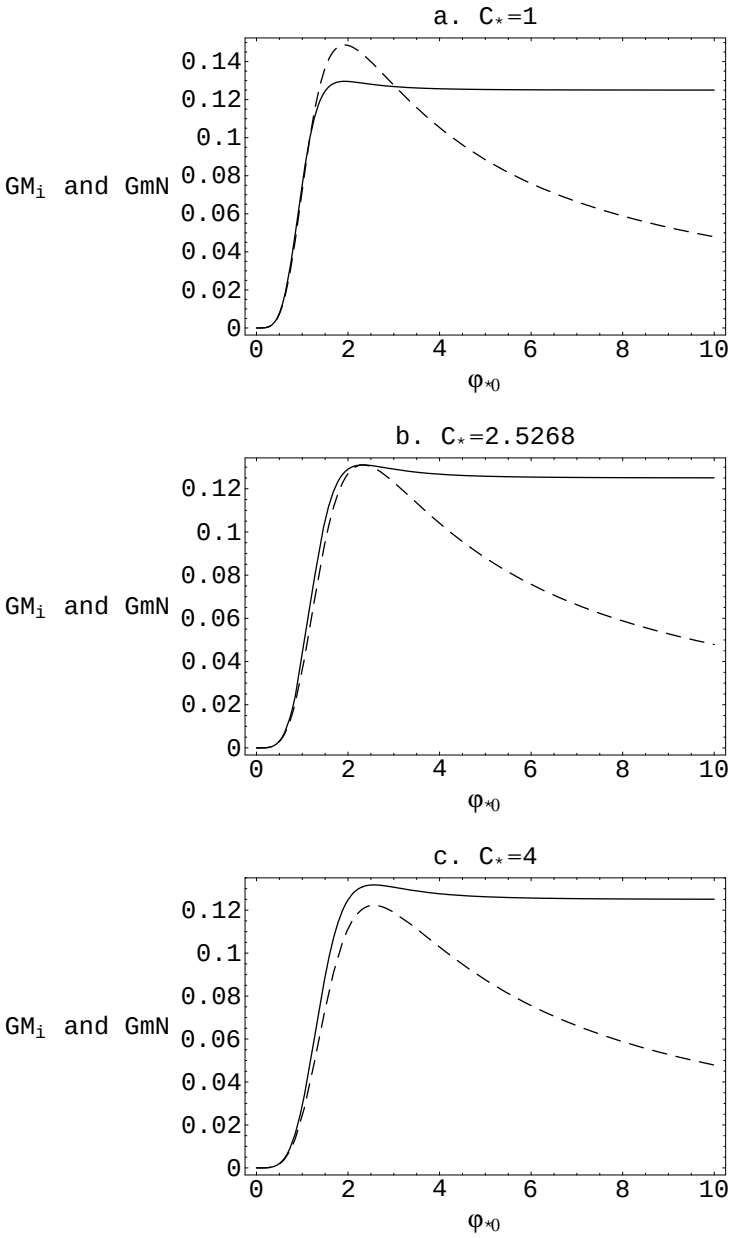

Figure 1: Boson star mass M_i in units of G^{-1} (solid line) and particle number N in units of $(Gm)^{-1}$ (broken line) as functions of φ_{*0} for different values of C_*: (a) $C_* = 1$. (b) $C_* = 2.5268$. (c) $C_* = 4$.

For small φ_{*0}, the mass and the partcle number are given approximately by

$$M_i \approx mN \approx \frac{\varphi_{*0}^4}{8GC_*} \,. \tag{51}$$

Both maxima of the mass and particle number as functions of φ_{*0} are located at $\varphi_{*0} = \varphi_{*0m}$. φ_{*0m} satisfies the following equation:

$$6C_*\varphi_{*0m}^2 - 4\varphi_{*0m}^4 - \varphi_{*0m}^6 + \left(-2C_* + 4\varphi_{*0m}^2 + 3\varphi_{*0m}^4\right) \ln\left(1 + \varphi_{*0m}^2\right) = 0 \,. \tag{52}$$

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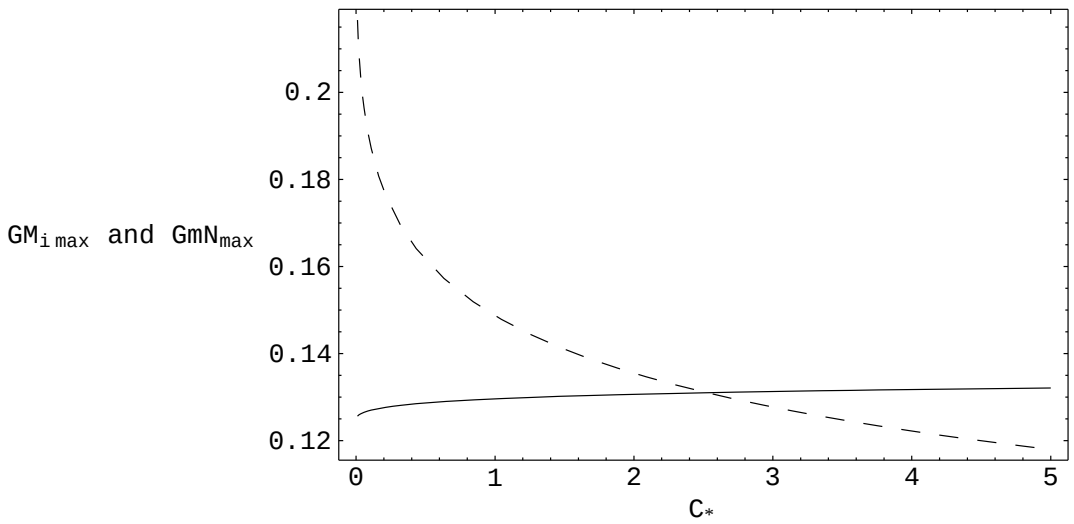

Figure 2: The value of maximum intrinsic mass $M_{i\ max}$ in units of G^{-1} (solid line) and The value of maximum particle number N_{max} in units of $(Gm)^{-1}$ (broken line) as functions of $C*$.

Unfortunately, the solution of this equation cannot be expressed by the usual mathematical functions. For small C_*, we can solve the equation (52) approximately and get $\varphi_{*0m} \approx (24C_*)^{1/6}$.

The maximum values of M_i and N, which are denoted by $M_{i\ max}$ and N_{max} respectively, is shown in Fig. 2 as functions of C_*. It is worth noting that the value of maximum BTZ mass ($M_{o\ max} = M_{i\ max} - \frac{1}{8G}$) is positive for all finite values of C_*. In particular, in the limit $C_* = 0$, the value of $M_{o\ max}$ vanishes. The behavior of $M_{o\ max}$ for small C_* is given by $M_{o\ max} \approx C_*^{2/3}/(32\sqrt[3]{3}G)$. The value of N_{max} in the limit $C_* = 0$ is $1/(4Gm)$.

The binding energy $E_{b\ max} = M_{i\ max} - mN_{max}$ for the maximum case is negative for sufficiently small C_*, as seen from Fig. 2. This implies that the boson star solution for small C_* is energetically stable. When the value of C_* is larger than a critical value $C_{*\ crit}$, $E_{b\ max}$ turns to be positive. We find that the critical value $C_{*\ crit}$ is

$$C_{*\ crit} = \frac{16}{243}\frac{1598 + 434\sqrt{13} - \left(1242 + 351\sqrt{13}\right)\ln\frac{4+\sqrt{13}}{3}}{10 + 4\sqrt{13} - 3\ln\frac{4+\sqrt{13}}{3}} \approx 2.5268\,. \qquad (53)$$

To see the meaning of the critical value, we plot the dependence of binding energy $E_b = M_i - mN$ on the particle number N for a fixed C_*. Fig. 3 shows the so-called bifurcation diagrams [6]. The cuspoidal point corresponds to the boson star with the mass $M_{i\ max}$ and the particle number N_{max}. This point represents an absolute stable solution [6]. When $C_* > C_{*\ crit}$, however, the cuspoidal point is not a global minimum of the binding energy. In this case, the boson star is not stable because the binding energy is positive for all finite N.[6]

[6]Notice also that the point corresponding to $N = 0$ and $E_b = 0$ is always a minimum point, because $E_b \approx +\varphi_{*0}^6/(24GC_*)$ for small φ_{*0}.

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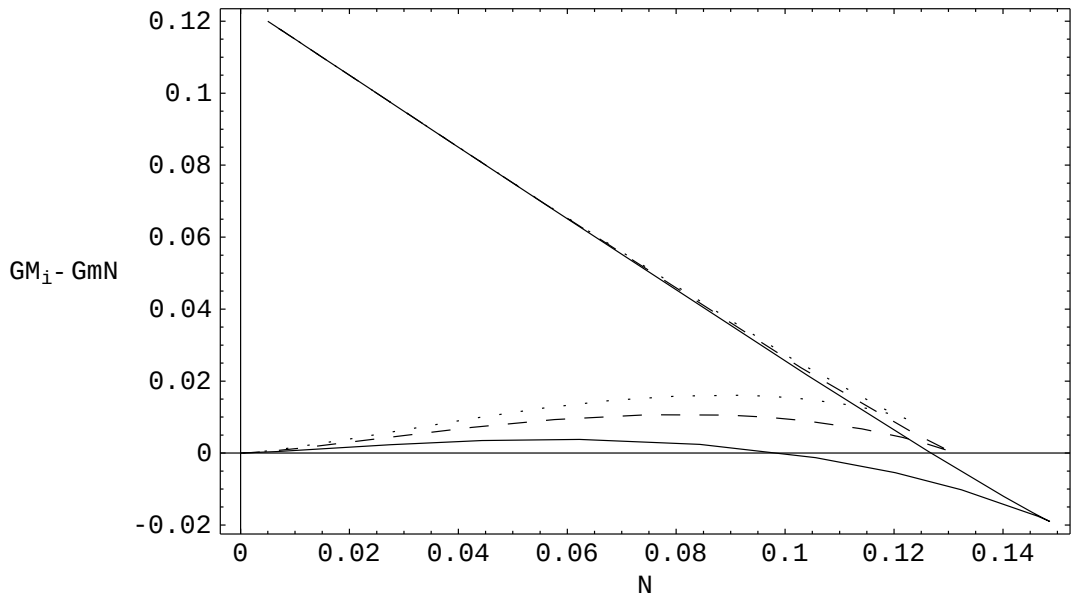

Figure 3: Binding energy E_b (in units of G^{-1}) as a function of N (in units of $(Gm)^{-1}$) for $C_* = 1$ (solid line), $C_{*\ crit}(\approx 2.5268)$ (broken line), and 4 (dotted line).

Lastly, we discuss the size of the boson star. We define the radius of the boson star by the radius of the boundary of the interior and exterior solutions, where $\varphi_* = 0$. The radius of the boson star R_b is derived from Eqs. (40) and (42) with the definition (36). We find

$$\begin{aligned} R_b^2 &= \frac{\Lambda}{m^2}\left[\frac{C_*\left(\varphi_{*0}^2+1\right)}{\left(\frac{1}{2}\varphi_{*0}^4+C_*\right)^2}+\frac{8GM_o}{C_*}\right] \\ &= \frac{\Lambda}{m^2}\frac{1}{\left(\frac{1}{2}\varphi_{*0}^4+C_*\right)^2}\left[\frac{1}{2}\varphi_{*0}^4+\varphi_{*0}^2-\left(\varphi_{*0}^2+1\right)\ln\left(\varphi_{*0}^2+1\right)+C_*\varphi_{*0}^2\right]. \end{aligned} \tag{54}$$

The radius of the boson star with maximum mass as a function of C_* is plotted in Fig. 4. For small C_*, R_b of the boson star with maximum mass behaves as $R_b \approx \sqrt{\frac{\Lambda}{m^2}}\frac{1}{3^{2/3}C_*^{1/6}}$.

The profile of the boson field can also be solved in the same way. That is given by

$$R^2 = \frac{\Lambda}{m^2}\frac{1}{\left(\frac{1}{2}\varphi_{*0}^4+C_*\right)^2}\left\{\left[\frac{1}{2}\left(\varphi_{*0}^2+1\right)+\frac{\frac{1}{2}+C_*}{\varphi_*^2+1}\right]\left(\varphi_{*0}^2-\varphi_*^2\right)+\left(\varphi_{*0}^2+1\right)\ln\frac{\varphi_*^2+1}{\varphi_{*0}^2+1}\right\}. \tag{55}$$

Of course R_b (54) is also obtained from this equation by setting $\varphi_* = 0$. One can see from (55) that α has no singular behavior, because $\alpha^2 = \tilde{\omega}^2/\left(\varphi_*^2+1\right)$. The profiles of the boson field φ_*^2 for specific values of C_* are shown in Fig. 5.

To summarize, we have obtained an exact solution describing a boson star with very large self-coupling constant in $(2+1)$ dimensions. There is a critical value for $C_* = (\lambda/(8\pi G m^4))C$, $C_{*\ crit} \approx 2.5268$. For $C_* > C_{*\ crit}$ the binding energy turns to be positive, the boson star configuration is not energetically favorable.

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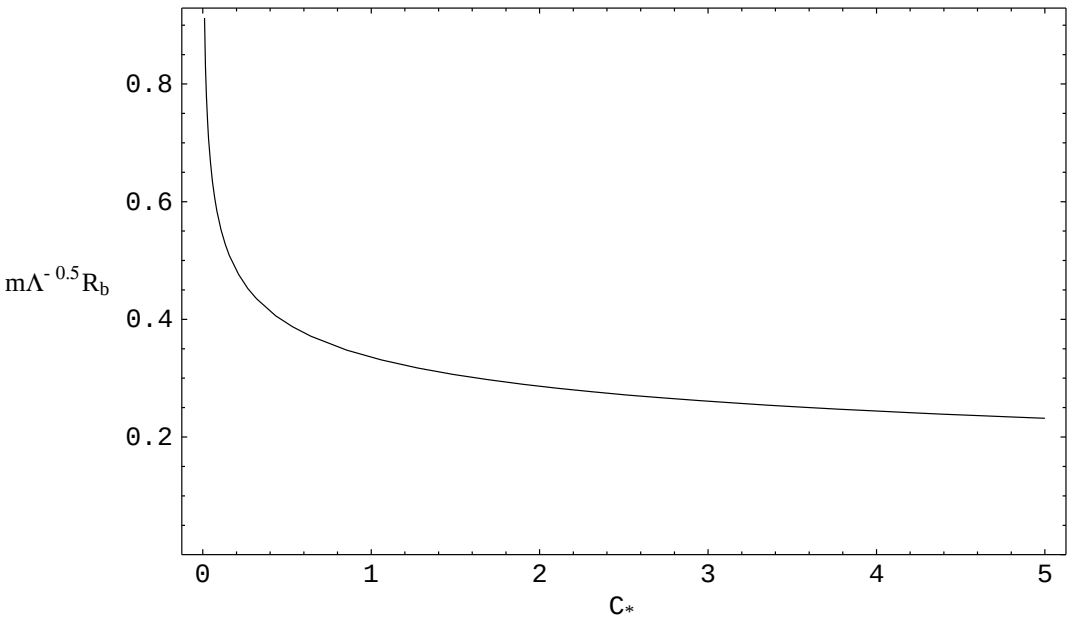

Figure 4: The radius of the boson star R_b (in units of $\sqrt{\Lambda/m^2}$) with maximum mass $M_{i\ max}$ as a function of C_*.

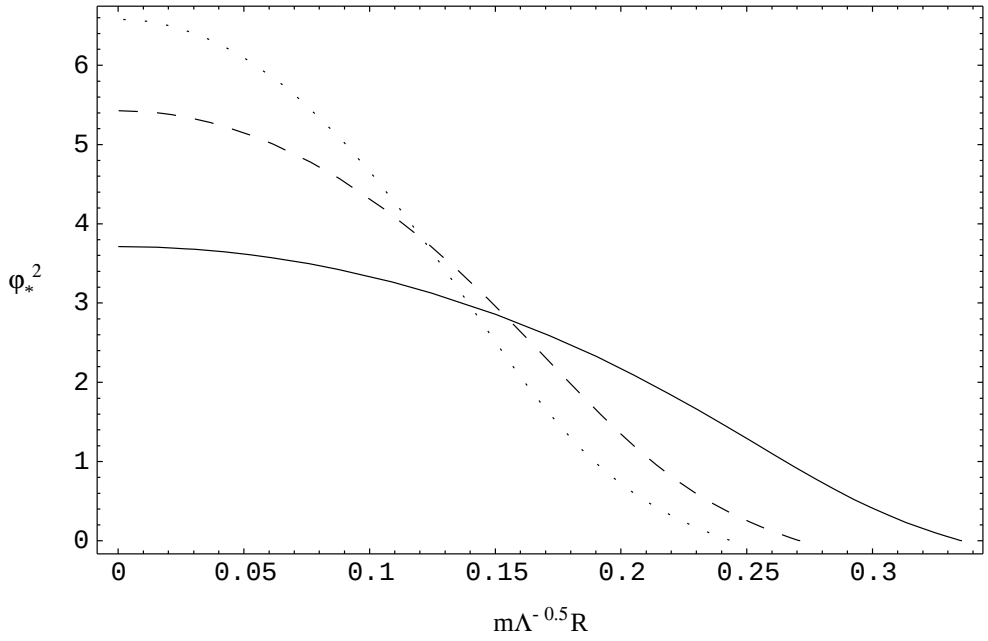

Figure 5: The behavior of the scalar field φ_*^2 as a function of R (in units of $\sqrt{\Lambda/m^2}$) for a boson star with maximum mass $M_{i\ max}$ for $C_* = 1$ (solid line), $C_{*\ crit} (\approx 2.5268)$ (broken line), and 4 (dotted line).

Explicit study of stability of the boson star under linear perturbations and pulsations [1, 2] will be discussed in future publications. The analysis of the boson star with an arbitrary, *actually finite* value of the self-coupling is of much interest and will be necessary.

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The rotation of the boson star[7] is easily incorporated in the $(2+1)$ dimensional model. The rigorous result on a nonrotating boson star in this paper is a basis of the study of the rotatinge case.

More general cases including such as a $|\varphi|^6$ coupling may exhibit more complicated results,but the analysis of them can be carried out in the same manner as in this paper. We expect that our analysis can also be easily extended to the solutions for fermion-boson stars in three dimensions. It will be the subject to be studied elsewhere.

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